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Applications of Phase Plane Analysis of a Liénard System to Positive Solutions of Schrödinger Equations

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1. INTRODUCTION

We consider the semilinear elliptic equation

$$\Delta u + f(x, u) = 0, \quad x \in \Omega, \quad (1)$$

where Ω is an exterior domain of \mathbb{R}^N with $N \geq 3$, that is, $G_a = \{x \in \mathbb{R}^N: |x| > a\} \subset \Omega$ for some $a > 0$. Throughout this paper, we assume that $f(x, u)$ is nonnegative and locally Hölder continuous with exponent $\alpha \in (0, 1)$ in $\overline{M} \times \overline{J}$ for every bounded domain $M \subset \mathbb{R}$ and for every bounded interval $J \subset \mathbb{R}$.

It is very famous that de Broglie's wave function

$$\psi(x, t) = \exp\left(-\frac{iEt}{\hbar}\right) v(x)$$

is a solution of the Schrödinger equation for a free particle of mass m , momentum p and kinetic energy E :

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta \psi,$$

where $\hbar = h/2\pi$ (h is Planck's constant) and

$$v(x) = A \exp\left(\frac{i(p \cdot x)}{\hbar}\right).$$

This equation is generalized into the Schrödinger equation with the potential V and the nonlinearity

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta \psi + V(x)\psi - g(x, |\psi|)\psi.$$

If it has standing waves solutions of the form

$$\psi(x, t) = \exp\left(-\frac{iEt}{\hbar}\right) u(x),$$

then the function $u(x)$ must satisfy the elliptic equation

$$\Delta u + \frac{2m}{\hbar^2} (E - V(x))u + g(x, |u|)u = 0,$$

which is of the form (1). In quantum mechanics, such are called stationary Schrödinger equations.

The aim of this paper is to give sufficient conditions under which equation (1) has a positive solution in an exterior domain of \mathbb{R}^N .

For a bounded domain $M \subset \Omega$, let $C^{2+\alpha}(\overline{M})$ denote the usual Hölder space. For simplicity, $C_{\text{loc}}^{2+\alpha}(\Omega)$ is defined as the set of all functions $u: \Omega \rightarrow \mathbb{R}$ such that $u \in C_{\text{loc}}^{2+\alpha}(\overline{M})$ for every bounded domain $M \subset \Omega$. A function $u \in C_{\text{loc}}^{2+\alpha}(\Omega)$ is called a *solution* of (1) in Ω if it satisfies equation (1) at every point $x \in \Omega$. Similarly, a function $u \in C_{\text{loc}}^{2+\alpha}(\Omega)$ is called a *supersolution* (resp., *subsolution*) of (1) in Ω if it satisfies the inequality $\Delta u + f(x, u) \leq 0$ (resp., ≥ 0) at every point $x \in \Omega$.

A typical example of (1) is the Emden-Fowler equation

$$\Delta u + p(x)u^\gamma = 0, \quad x \in \Omega,$$

where $p(x)$ is nonnegative and locally Hölder continuous in Ω and γ is a positive number. From this fact, equation (1) is often discussed under a sublinear or a superlinear hypothesis. For instance, equation (1) is said to be *sublinear* (resp., *superlinear*) if there exists a γ with $0 < \gamma < 1$ (resp., $\gamma > 1$) such that $f(x, u)/u^\gamma$ is nonincreasing (resp., nondecreasing) in u for each fixed $r = |x| > 0$.

Many studies have been made on the existence of a positive solution of (1) in the linear case, the sublinear case and the superlinear case (see [2, 4, 5, 6, 7]). In this paper, we intend to examine another case in addition to these cases. For example, consider the case that

$$f(x, u) = p(x) \left(u + \frac{u}{4(\log u)^2} \right) \quad (2)$$

for all sufficiently small u . Then equation (1) is neither sublinear nor superlinear (of course, equation (1) is not linear). In fact, differentiating $f(x, u)/u^\gamma$, we have

$$\frac{d}{du} \left(\frac{f(x, u)}{u^\gamma} \right) = \frac{p(x)}{u^\gamma} \left\{ (1 - \beta) + \frac{1 - \beta - 2/\log u}{4(\log u)^2} \right\}.$$

Hence, if $0 < \gamma < 1$ (resp., $\gamma > 1$), then $f(x, u)/u^\gamma$ is increasing (resp., decreasing) for $u > 0$ sufficiently small. In the case (2), for any $k > 1$, there exists a positive interval I such that

$$p(x)u < f(x, u) < kp(x)u$$

for all $x \in \Omega$ and $u \in I$. Hence, from this point of view, we may say that equation (1) is *almost linear* in such cases as (2).

For sublinear Schrödinger equations, Swanson [7, Theorem 2.4] gave the following sufficient condition for the existence of a positive solution under the assumption that

$$0 \leq f(x, u) \leq u\varphi(|x|, u) \quad (3)$$

for all $x \in \Omega$ and $u > 0$, where $\varphi(r, u)$ is locally Hölder continuous with exponent $\alpha \in (0, 1)$ and nonincreasing in u for each fixed $r > 0$.

Theorem A. *Under the assumption (3), equation (1) has a positive solution in an exterior domain if*

$$\int_0^\infty r\varphi(r, c)dr < \infty \quad (4)$$

for some $c > 0$.

Consider the case that $f(x, u) = u/4|x|^\beta$ with $\beta \geq 2$. Then assumption (3) is satisfied with $\varphi(r, u) = 1/4r^\beta$. Since

$$\int_0^\infty r\varphi(r, c)dr = \int_0^\infty \frac{1}{4r^{\beta-1}}dr,$$

for any $c > 0$, condition (4) is satisfied if $\beta > 2$, but it does not hold if $\beta = 2$. Hence, Theorem A is inapplicable to the case $\beta = 2$. However, the equation

$$\Delta u + \frac{u}{4|x|^2} = 0$$

has a positive solution, because its radial solutions are represented as the form of

$$u(x) = \begin{cases} (K_1 + K_2 \log |x|)|x|^{-1/2} & \text{if } N = 3, \\ K_3|x|^z + K_4|x|^{2-N-z} & \text{if } N \geq 4, \end{cases}$$

where K_i ($i = 1, 2, 3, 4$) are arbitrary constants and z is the root of $z^2 + (N-2)z + 1/4 = 0$.

Assumption (3) is not compatible with the superlinear case and the almost linear case. Hence, instead of (3), we assume that

$$0 \leq f(x, u) \leq \frac{h(u)}{|x|^2} \quad (5)$$

for all $x \in \Omega$ and $u \geq 0$, where $h(u)$ is locally Lipschitz continuous and positive for $u > 0$, and $h(0) = 0$. We also prepare the following notation to present a theorem which can be applied to these cases. Write

$$L_1(u) = 1, \quad L_{n+1}(u) = L_n(u) l_n(u), \quad n = 1, 2, \dots,$$

where

$$l_1(u) = 2|\log u|, \quad l_{n+1}(u) = \log\{l_n(u)\},$$

and set

$$S_n(u) = \sum_{k=1}^n \frac{1}{\{L_k(u)\}^2}.$$

Define $e_0 = 1$ and $e_n = \exp(e_{n-1})$. Then we have

$$l_{n+1}(u) = \log\{l_n(u)\} > 0 \quad \text{for } 0 < u < 1/\sqrt{e_n},$$

and therefore, the function sequences $\{L_n(u)\}$, $\{l_n(u)\}$ and $\{S_n(u)\}$ are well-defined for $u > 0$ sufficiently small. To take some concrete forms of $S_n(u)$, for $u > 0$ sufficiently small,

$$S_1(u) = 1, \quad S_2(u) = 1 + \frac{1}{4(\log u)^2}, \quad S_3(u) = 1 + \frac{1}{4(\log u)^2} + \frac{1}{4(\log u)^2 (\log(2|\log u|))^2},$$

and so on.

Our main result is stated in the following:

Theorem 1. Assume (5) and suppose that there exists a positive integer n such that

$$\frac{h(u)}{u} \leq \frac{(N-2)^2}{4} S_n(u) \quad (6)$$

for all $u > 0$ sufficiently small. Then equation (1) has a positive solution $u(x)$ in an exterior domain with $\lim_{|x| \rightarrow \infty} u(x) = 0$.

2. A SUPERSOLUTION AND A SUBSOLUTION

We will prove the main result by use of the so-called “supersolution-subsolution” method. The lemma below yields from a result of Noussair and Swanson [5, Theorem 3.3].

Lemma 2. If there exists a positive supersolution \bar{u} of (1) and a positive subsolution \underline{u} of (1) in G_b such that $\underline{u}(x) \leq \bar{u}(x)$ for all $x \in G_b \cup C_b$, where $b \geq a$ and $C_b = \{x \in \mathbb{R}^N: |x| = b\}$, then equation (1) has at least one solution u satisfying $u(x) = \bar{u}(x)$ on C_b and $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$ through G_b .

To apply Lemma 2, we have to find a suitable positive supersolution of (1) and a positive subsolution of (1) which is not greater than the supersolution. For this purpose, we consider the nonlinear differential equation

$$\frac{d^2}{dr^2}w + \frac{N-1}{r} \frac{d}{dr}w + \frac{1}{r^2}g(w) = 0, \quad r > a, \quad (7)$$

where $g(w)$ satisfies a suitable smoothness condition for the uniqueness of solutions of the initial value problem and the signum condition

$$wg(w) > 0 \quad \text{if } w \neq 0. \quad (8)$$

Then we have the following nonoscillation theorem for equation (7).

Lemma 3. Assume (8). If there exists a positive integer n such that

$$\frac{g(w)}{w} \leq \frac{(N-2)^2}{4} S_n(|w|) \quad (9)$$

for $w > 0$ or $w < 0$, $|w|$ sufficiently small, then all nontrivial solutions of (7) are nonoscillatory.

Proof. Using phase plane analysis of a Liénard system, Sugie *et al.* [10, Lemma 3.2] proved that under the assumption (8), all nontrivial solutions of the equation

$$\frac{d^2}{dr^2}w + \frac{2}{r} \frac{d}{dr}w + \frac{1}{r^2}g(w) = 0 \quad (10)$$

are nonoscillatory if

$$\frac{g(w)}{w} \leq \frac{1}{4} S_n(|w|) \quad (11)$$

for $w > 0$ or $w < 0$, $|w|$ sufficiently small. Hence, the lemma is true for $N = 3$.

Suppose that $N \geq 4$. Let

$$\tau = (N - 2)r^{N-2} \quad \text{and} \quad v(\tau) = w(r).$$

Then equation (7) becomes

$$\frac{d^2}{d\tau^2}v + \frac{2}{\tau} \frac{d}{d\tau}v + \frac{1}{\tau^2}g^*(v) = 0,$$

where $g^*(v) = g(v)/(N - 2)^2$. This equation has the form of (10). It follows from (9) that

$$\frac{g^*(w)}{w} = \frac{g(w)}{(N - 2)^2 w} \leq \frac{1}{4}S_n(|w|)$$

for $w > 0$ or $w < 0$, $|w|$ sufficiently small, that is, (11) is satisfied with $g(w) = g^*(w)$. Hence, by Lemma 3.2 in [10] again, we see that all nontrivial solutions of (7) are nonoscillatory in the case $N \geq 4$. \square

By virtue of Lemma 3, we can choose a solution of (7) which is eventually positive. In the next section, we will show that the positive solution is a supersolution of (1). To get a positive subsolution of (1), we need to estimate the asymptotic behavior of positive solutions of (7) as follows.

Lemma 4. *Assume (8) and (9). Then there exist a positive number $b \geq a$ and a positive solution $w(r)$ of (7) such that $\lim_{r \rightarrow \infty} w(r) = 0$*

$$b^{N-2}w(b) \leq r^{N-2}w(r) \quad \text{for } r \geq b.$$

Proof. From Lemma 3 we see that equation (7) has a positive solution. Let $w(r)$ be the positive solution. Then there exists a $b \geq a$ such that

$$w(r) > 0 \quad \text{for } r \geq b.$$

The change of variables $r = e^s$ and $w(r) = \xi(s)$ transforms equation (7) into the Liénard system

$$\begin{aligned} \frac{d}{ds}\xi &= \eta - (N - 2)\xi, \\ \frac{d}{ds}\eta &= -g(\xi). \end{aligned} \tag{12}$$

Let $(\xi(s), \eta(s))$ be the solution of (12) corresponding to $w(r)$. Then we have

$$\xi(s) > 0 \quad \text{for } s \geq \log b. \tag{13}$$

By (8) we obtain

$$\frac{d}{ds}\eta(s) < 0 \quad \text{for } s \geq \log b. \tag{14}$$

It is well known that the zero solution of (12) is globally asymptotically stable (for example, see [1, 3, 8]). Hence, we conclude that the solution $(\xi(s), \eta(s))$ tends to the origin as $s \rightarrow \infty$. This means that $w(r)$ approaches the zero as $r \rightarrow \infty$.

We will show that $\eta(s) \geq 0$ for $s \geq \log b$. Suppose that $\eta(s_0) < 0$ for some $s_0 \geq \log b$. Then, by (12)–(14) we have

$$\frac{d}{ds}\xi(s) < \frac{d}{ds}\xi(s) + (N-2)\xi(s) = \eta(s) \leq \eta(s_0)$$

for $s \geq s_0$. Integrate this inequality from s_0 to s to obtain

$$\xi(s) < \xi(s_0) + \eta(s_0)(s - s_0) \rightarrow -\infty \quad \text{as } s \rightarrow \infty.$$

This is a contradiction to (13).

Since $\eta(s) \geq 0$ for $s \geq \log b$, we see that

$$\frac{d}{ds}\xi(s) \geq -(N-2)\xi(s) \quad \text{for } s \geq \log b.$$

Hence, integrating the both sides, we have

$$b^{N-2}\xi(\log b) \leq e^{(N-2)s}\xi(s) \quad \text{for } s \geq \log b,$$

namely, $b^{N-2}w(b) \leq r^{N-2}w(r)$ for $r \geq b$. Thus, the lemma is proved. \square

We are now ready to prove the main theorem.

3. PROOF OF THE MAIN THEOREM

Consider the nonlinear differential equation

$$\frac{d^2}{dr^2}w + \frac{N-1}{r} \frac{d}{dr}w + \frac{1}{r^2}h^*(w) = 0, \quad r \geq a, \quad (15)$$

where a is the number given in (1) and

$$h^*(w) = \begin{cases} h(w) & \text{for } w \geq 0, \\ -h(-w) & \text{for } w < 0. \end{cases}$$

Then, from assumption (5) we see that $h^*(w)$ satisfies the signum condition (8), and therefore, equation (15) is in the type of (7). Also, by condition (6) we have

$$\frac{h^*(w)}{w} \leq \frac{1}{4}S_n(|w|)$$

for $w > 0$ and $w < 0$, $|w|$ sufficiently small. Hence, from Lemma 3 we conclude that all nontrivial solutions of (15) are nonoscillatory. For this reason, we can choose a solution $w(r)$ which is positive for all $r \geq b$ with some $b \geq a$ (we may regard b as the positive

number in Lemma 4). As in the proof of Lemma 4, we can show that $w(r)$ approaches the zero as r tends to ∞ . Note that $w(r)$ is also a positive solution of the equation

$$\frac{d^2}{dr^2}w + \frac{N-1}{r} \frac{d}{dr}w + \frac{1}{r^2}h(w) = 0.$$

Let \bar{u} be the function defined in G_b by $\bar{u}(x) = w(r)$, $r = |x| \geq b$. Then, by assumption (5) we obtain

$$\begin{aligned} \Delta \bar{u}(x) + f(x, \bar{u}(x)) &= \frac{d^2}{dr^2}w(r) + \frac{N-1}{r} \frac{d}{dr}w(r) + f(x, w(r)) \\ &\leq \frac{d^2}{dr^2}w(r) + \frac{N-1}{r} \frac{d}{dr}w(r) + \frac{1}{|x|^2}h(w(r)) \\ &= \frac{d^2}{dr^2}w(r) + \frac{N-1}{r} \frac{d}{dr}w(r) + \frac{1}{r^2}h(w(r)) = 0 \end{aligned}$$

Hence, \bar{u} is a supersolution of (1) in G_b . We next denote $\underline{u}(x) = b^{N-2}w(b)/|x|^{N-2}$ for $|x| \geq b$. Then, since $f(x, u)$ is nonnegative, we get

$$\begin{aligned} \Delta \underline{u}(x) + f(x, \underline{u}(x)) &\geq \frac{d^2}{dr^2} \left(\frac{b^{N-2}w(b)}{r^{N-2}} \right) + \frac{N-1}{r} \frac{d}{dr} \left(\frac{b^{N-2}w(b)}{r^{N-2}} \right) \\ &= \frac{(N-2)(N-1)b^{N-2}w(b)}{r^N} - \frac{N-1}{r} \frac{(N-2)b^{N-2}w(b)}{r^{N-1}} = 0. \end{aligned}$$

This means that $\underline{u}(x)$ is a subsolution of (1) in G_b .

From Lemma 4 we see that

$$\underline{u}(x) = \frac{b^{N-2}w(b)}{|x|^{N-2}} = \frac{b^{N-2}w(b)}{r^{N-2}} \leq w(r) = \bar{u}(x)$$

for $|x| \geq b$. Hence, by means of Lemma 2, we conclude that there exists a positive solution $u(x)$ of (1) satisfying $\underline{u}(x) = u(x) = \bar{u}(x)$ on C_b and $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$ through G_b . Since $w(r)$ tends to the zero as $r \rightarrow \infty$, the positive solution $u(x)$ also tends to the zero as $|x| \rightarrow \infty$. This completes the proof. \square

4. DISCUSSION

To illustrate the main theorem, we will give some examples which are the almost linear case. One cannot apply previous results on the existence of a positive solution to those examples. For brevity, we define the function $\phi(u; \lambda)$ by $\phi(0; \lambda) = 0$ for any $\lambda \geq 0$ and

$$\phi(u; \lambda) = \begin{cases} u + \frac{\lambda u}{(\log |u|)^2} & \text{for } 0 < u \leq \frac{1}{e}, \\ (3\lambda + 1)u - \frac{2\lambda}{e} & \text{for } u > \frac{1}{e}. \end{cases}$$

Then it is easy to check that $\phi(u; \lambda)$ is continuous for $u \geq 0$ and is continuously differentiable for $u > 0$.

We first consider the elliptic equation

$$\Delta u + p(x)\phi(u; 1/4) = 0 \quad (16)$$

in an exterior domain Ω of \mathbb{R}^N with $N \geq 3$. Let

$$f(x, u) = p(x)\phi(u; 1/4) \quad \text{and} \quad h(u) = \frac{(N-2)^2}{4}\phi(u; 1/4).$$

Then condition (5) holds and condition (6) is satisfied with $n = 2$. Hence, as a direct consequence of Theorem 1, we have the following result.

Example 5. If

$$0 \leq p(x) \leq \frac{(N-2)^2}{4|x|^2}$$

for $x \in \Omega$, then equation (16) has a decaying positive solution.

Let us take another example to show how sharp Theorem 1 is. For this purpose, we restrict $p(x)/|x|^2$ to any constant.

Example 6. Consider the equation with two parameters

$$\Delta u + \frac{\mu}{|x|^2}\phi(u; \lambda) = 0 \quad (17)$$

instead of (16). Then, from Theorem 1 we have the following conclusions:

- (i) if $0 \leq \mu < (N-2)^2/4$, then equation (17) has a decaying positive solution for all $\lambda \geq 0$;
- (ii) if $\mu = (N-2)^2/4$, then equation (17) has a decaying positive solution for $0 \leq \lambda \leq 1/4$.

Proof. Let $f(x, u) = \mu\phi(u; \lambda)/|x|^2$ and $h(u) = \mu\phi(u; \lambda)$. Since λ and μ are nonnegative, condition (5) is satisfied. Hence, it is enough to check that condition (6) holds for $u > 0$ sufficiently small. If $\lambda = 0$, then $h(u)/u = \mu \leq (N-2)^2/4$ for all $u > 0$, that is, condition (6) is satisfied with $n = 1$. We assume that λ is positive.

(i) We can choose an $\varepsilon_0 > 0$ so small that $\mu(1 + \varepsilon_0) < (N-2)^2/4$. For any $\lambda > 0$, we see that

$$\frac{h(u)}{u} = \mu \left(1 + \frac{\lambda}{(\log u)^2} \right) < \mu(1 + \varepsilon_0) < \frac{(N-2)^2}{4}$$

for $0 < u < \exp(-\sqrt{\lambda/\varepsilon_0})$. Hence, condition (6) is satisfied with $n = 1$.

(ii) In this case, we have

$$\frac{h(u)}{u} = \mu \left(1 + \frac{\lambda}{(\log u)^2} \right) \leq \frac{(N-2)^2}{4} \left(1 + \frac{1}{4(\log u)^2} \right)$$

for u sufficiently small, namely, condition (6) with $n = 2$. □

Recently, by use of phase plane analysis of a Liénard system, Sugie *et al.* [9, Lemma 4.4] gave an oscillation theorem for equation (10) under the assumption (8) as follows.

Theorem B. Assume (8) and suppose that there exists a λ with $\lambda > 1/4$ satisfying

$$\frac{g(w)}{w} \geq \frac{1}{4} + \frac{\lambda}{(2 \log |w|)^2} \quad (18)$$

for $|w|$ sufficiently small. Then all nontrivial solutions of (10) are oscillatory.

To compare with the conclusion (ii) of Example 6, we consider the equation

$$\Delta u + \frac{(N-2)^2}{4|x|^2} \phi^*(u; \lambda) = 0, \quad (19)$$

where

$$\phi^*(u; \lambda) = \begin{cases} \phi(u; \lambda) & \text{for } u \geq 0, \\ -\phi(-u; \lambda) & \text{for } u < 0. \end{cases}$$

It is clear that $\phi^*(u; \lambda)$ is odd, and therefore, it satisfies the signum condition (8). As shown in Sections 2 and 3, the change of variables

$$v(\tau) = w(r) = u(x), \quad r = |x| \quad \text{and} \quad \tau = (N-2)r^{N-2}$$

reduces equation (19) to

$$\frac{d^2}{d\tau^2} v + \frac{2}{\tau} \frac{d}{d\tau} v + \frac{1}{4\tau^2} \phi^*(v; \lambda) = 0.$$

This is of the form (10). Since

$$\frac{\phi^*(v; \lambda)}{4v} = \frac{1}{4} + \frac{\lambda}{(2 \log |v|)^2}$$

for $|v|$ sufficiently small, from Theorem B it turns out that if $\lambda > 1/4$, then equation (19) fails to have positive radial solutions. Hence, together with the second conclusion in Example 6, we see that equation (19) has a positive radial solution if and only if $\lambda \leq 1/4$.

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